Pressures and Energies of Vacuum in a Magnetic Field. Differences and Analogies with Casimir Effect

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Abstract. We study the electron-positron vacuum in a strong magnetic field B in parallel with Casimir effect. Starting from the energy eigenvalues, anisotropic pressures are obtained in both magnetic field and Casimir cases. In the first case the pressure transversal to the field B is negative due to the effect of vacuum magnetization, whereas along B an usual positive pressure arises. Similarly, in addition to the usual negative Casimir pressure perpendicular to the plates, the existence of a positive pressure along the plates is predicted. By assuming regions of the universe having random orientation of the lines of force, cosmological consequences are discussed in the magnetic field case.

Keywords: magnetic field, QED vacuum, astrophysics, cosmology
PACS: 95.30.Cq, 98.80.Cq, 12.20.Ds

INTRODUCTION

There is some similarity among the effects of an external field and certain boundary conditions. For instance, this is the case of virtual electron-positron pairs of vacuum under the action of an external constant magnetic field, and the virtual photon field modes inside the cavity formed by two parallel metallic plates (Casimir effect [1],[2]. For more recent developments see [3]). Both problems bear some interesting analogy, the metallic plates keeping bounded the motion of virtual photons perpendicular to the plates whereas the external field bound electrons and positrons to Landau quantum states in the plane perpendicular to it. The comparative study of these two QED vacua (in an independent way, but following a common method) is interesting in itself, exhibiting the property of anisotropic pressures, having negative values in some directions, as well as negative energy density.

We assume the magnetic field characterized by the microscopic magnetic field pseudo-vector $B_i = \varepsilon_{ijk}\mathscr{F}_{jk}$ (where $\mathscr{F}_{jk}$ is the spatial part of the electromagnetic field tensor $\mathscr{F}_{\mu\nu}$), leading to the breaking of the spatial symmetry: the spinor wavefunctions and spectrum of charged particles having axial symmetry [4], their motion being bounded perpendicularly to B (this is valid also for the zero point modes of vacuum). In the case of the the well-known Casimir effect it is produced when two parallel metallic plates are placed in vacuum, leading to the vanishing of the electric field component tangential to the plates. The plates having characteristic diameter a and separated by a distance d, with $a >> d$ (we shall assume in what follows $a \to \infty$). Only modes whose wave vector components perpendicular to the cavity are integer multiples of

$k_{03} = \pi/d$, are allowed inside it. This makes the zero point electromagnetic modes inside the box axially symmetric in momentum space. In both problems there is a quantity characterizing the symmetry breaking, (and the extension of the wave functions in some direction). These quantities are respectively, the pseudo-vector B_i, determining $\lambda_L(B)$, and the basic vector momentum $\mathscr{P}_i = \mathscr{P}\delta_{3i}$, perpendicular to the plates (which are taken parallel to the x_1, x_2 plane). Here $\mathscr{P} = \hbar k_{03}$ and the length d characterizes the extension of the wave function perpendicular to the plates.

In the magnetic field case [4], the solution of the Dirac equation for an electron (or positron) in presence of an external magnetic field B_j for, say, $j = 3$, leads to the energy eigenvalues,

$$\varepsilon_n = \sqrt{c^2 p_3^2 + m^2 c^4 + 2e\hbar cBn}, \tag{1}$$

where $n = 0, 1, 2...$ are the Landau quantum numbers, p_3 is the momentum component along the magnetic field $\mathbf{B}$ and m is the electron mass. The breaking of the spatial symmetry due to the magnetic field is manifested in the spectrum as an harmonic oscillator-like quantization of the energy in the direction perpendicular to the field.

In the Casimir effect the motion of virtual photons perpendicular to the plates is bounded and we have the photon energy eigenvalues,

$$\varepsilon_s = c\sqrt{p_1^2 + p_2^2 + (\mathscr{P}s)^2}. \tag{2}$$

Due to the breaking of the rotational symmetry in momentum space inside the cavity, only vacuum modes of discrete momentum $p_3 = \mathscr{P}s$ where $s = 0, \pm 1, \pm 2, ...$ are allowed. After taking the sum of all these modes and subtracting the divergent part from it [2], one gets a finite negative term dependent on $\mathscr{P}$, which is the vacuum energy. From this, it was shown by Casimir [1], [2] that a negative pressure appears in between the plates and perpendicular to them.

Starting from the analogy between (2) and (1), which have as a common property a discrete quantization of some of its momentum components, it would be interesting to investigate in parallel both physical systems, specially since a negative pressure arises as due to the zero point electron-positron vacuum energy in an external constant magnetic field.

VACUUM ZERO POINT ENERGIES

The electron-positron zero point vacuum energy in an external electromagnetic field was obtained by Heisenberg and Euler [5]. For the case of a pure magnetic field, it contains the contribution coming from the virtual electron-positron pairs created and annihilated spontaneously in vacuum and interacting with the field B. In the one loop approximation, where no radiative corrections are considered, the renormalized vacuum energy density[1]

[1] We observe that the vacuum term regularization demands the addition of a negative infinite term proportional to B^2 which absorbs the classical energy term $B^2/8\pi$.

is given by

$$\Omega_{0e} = \frac{\alpha B^2}{8\pi^2}\int_0^\infty e^{-B_c x/B}\left[\frac{coth x}{x} - \frac{1}{x^2} - \frac{1}{3}\right]\frac{dx}{x}, \tag{3}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant and $B_c = m^2c^3/e\hbar = 4.41 \cdot 10^{13}$G is the QED critical magnetic field. As the quantity in squared brackets in (3) is negative, we have $\Omega_{0e} < 0$. [2]

The density of energy Ω_{0C} for the Casimir problem may be obtained directly from (2) by summing over s. This is mathematically equivalent to find the thermodynamic potential of radiation at a temperature $T_{Cas} = \hbar c/2d$, according to finite temperature quantum field theory methods [7]. One gets a finite term, dependent on d, and a four dimensional divergence independent of d. The finite energy density is

$$\Omega_{0C} = -\frac{\pi^2\hbar c}{720 d^4} = -\frac{c\mathscr{P}^4}{720\pi^2\hbar^3}. \tag{4}$$

Returning to (3), as fields currently achieved in laboratories are very small if compared with the critical field B_c, in the limit $B << B_c$ one can write,

$$\Omega_{0e} \approx -\frac{\alpha B^4}{360\pi^2 B_c^2} = -\frac{\pi^2\hbar c}{5760 b^4}, \tag{5}$$

where the characteristic parameter is $b(B) = \pi\lambda_L^2/\lambda_C$. Here λ_L is the magnetic wavelength defined previously and λ_C is the Compton wavelength $\lambda_C = \hbar/mc$. The energy density is then a function of the field dependent parameter $b(B)$. The expression for (5) looks similar to (4).

VACUUM PRESSURES

Quantum statistics at temperature T and chemical potential μ leads to quantum field theory in vacuum if the limit limit $T \to 0$, $\mu \to 0$ is taken (see e.g. Fradkin [7]; the functional average $<< .. >>$ becomes the quantum field average). This is because the contribution of observable particles, given by the statistical term $\Omega_s(T,\mu)$ in the expression for the total thermodynamic potential $\Omega = \Omega_s + \Omega_0$, vanishes in that limit. The remaining term, which is the contribution of virtual particles, leads to the zero point energy of vacuum Ω_0.

The energy-momentum tensor of matter is a diagonal one; spatial part contains the pressures and the time component is minus the internal energy density $-U$ (see [8]). We obtain then vacuum pressures as the limit $T \to 0$, $\mu \to 0$ of the expressions for spatial

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[2] We want to stress at this point that our aim is to study both Casimir and magnetic field problems as independent. Both problems were investigated together to find the magnetic permeability in Casimir vacuum in [6].

components of the matter energy-momentum tensor[3]. We present here the results for the Casimir effect and for the QED vacuum in a magnetic field.

Casimir effect

The energy-momentum tensor for the Casimir effect in vacuum may be obtained, also, directly by following a complete parallelism with the temperature case, since the four-momentum vector breaking the spatial symmetry has a discrete component $p_3 = \mathscr{P}n$, $n = 0, \pm 1, \pm 2...$ After quantum averaging, one has the anisotropic pressures

$$\mathscr{T}_3^3 = P_{C3} = \mathscr{P}\frac{\partial \Omega_{0C}}{\partial \mathscr{P}} - \Omega_{0C} = 3\Omega_{0C} = -\frac{\pi^2 \hbar c}{240 d^4} < 0 \tag{6}$$

which is the usual Casimir negative pressure [1] and

$$\mathscr{T}_\perp^C = P_{C\perp} = -\Omega_{0C} = \frac{\pi^2 \hbar c}{720 d^4} > 0 \tag{7}$$

which is a positive pressure acting parallel to the plates in the region inside them. This is a second Casimir force. (This *is not* the so-called lateral Casimir force reported in [9]). The combined action of both forces suggests a flow of QED vacuum out of the cavity inside the plates, as a fluid which is compressed by the attractive force exerted between them.

Magnetized vacuum

For matter in an external magnetic field, an anisotropy in the pressures occurs [8], [10]. The anisotropy is due to the arising of a negative transverse pressure, generated by an axial "force": the quantum analog of the Lorentz force, arising when the magnetic field acts on charged particles having non-zero spin [11], and leading to a magnetization parallel to **B**. For vacuum limit, according to ([8], [10]), the diagonal components of the energy-momentum tensor lead to a positive pressure $\mathscr{T}_3^{0e3} = P_{03} = -\Omega_{0e}$ along the magnetic field B, and to $\mathscr{T}_\perp^{0e} = P_{0\perp} = -\Omega_{0e} - B\mathscr{M}_{0e}$ in the direction perpendicular to the field. Here $\mathscr{M}_{0e} = -\partial\Omega_{0e}/\partial B$ is the vacuum magnetization, which is obtained from (3) as

$$\mathscr{M}_{0e} = -\frac{2\Omega_{0e}}{B} - \frac{\alpha B_c}{8\pi^2}\int_0^\infty e^{-B_c x/B}\left[\frac{\coth x}{x} - \frac{1}{x^2} - \frac{1}{3}\right] dx. \tag{8}$$

One can check easily that (8) is a positive quantity. Moreover, it has a non- linear dependence on the field B. Then it may be stated that the quantum vacuum has paramagnetic properties (and $\partial\mathscr{M}_{0e}/\partial B > 0$). In our present one-loop approximation we do

[3] A detailed discussion about the magnetized vacuum contribution to the matter energy momentum tensor and about the energy momentum tensor components for the Casimir effect will appear in a forthcoming paper.

not consider the spin-spin interaction between virtual particles, which would lead to a ferromagnetic behavior.

Concerning the transverse pressure $P_{0e\perp} = -\Omega_{0e} - B\mathscr{M}_{0e}$, we get

$$\mathscr{T}_{\perp}^{0e} = P_{0e\perp} = \Omega_{0e} + \frac{\alpha B_c B}{8\pi^2} \int_0^\infty e^{-B_c x/B} \left[\frac{\coth x}{x} - \frac{1}{x^2} - \frac{1}{3} \right] dx. \tag{9}$$

Both terms in (9) are negative, thus, $P_{0e\perp} < 0$, whereas along the field, the pressure $P_{03} = -\Omega_{0e}$ is positive. This leads to magnetostrictive effects for small as well as for high fields. Thus, QED vacuum in a magnetic field B is compressed perpendicular to it, and as the pressure is positive along B, it is stretched in that direction. This is reasonable to expect: the virtual electrons and positrons are constrained to bound states in the external field, but flows freely in both directions along the field. That motion of virtual particles can be interpreted as similar to the real electrons and positrons, describing "orbits" having a characteristic radius of order $\lambda = \sqrt{\hbar c/eB}$ in the plane orthogonal to B, but the system is degenerate with regard to the position of the center of the orbit. It must be stressed that the term $B\mathscr{M}_{0e}$ subtracted by $-\Omega_{0e}$ in $P_{0e\perp}$ is the statistical pressure due to the quantum version of the Lorentz force acting on particles (in the present case virtual) bearing a magnetic moment, which leads to $\mathscr{M}_{0e} > 0$ [11]. In the low energy limit $eB << m^2$ we have $P_{0e\perp} \approx 3\Omega_{0e} < 0$. It can be written

$$P_{0e\perp} \approx -\frac{\pi^2 \hbar c}{1920 b^4}, \tag{10}$$

For small B fields of order $10 - 10^3$G, $P_{0e\perp}$ is negligible as compared with the usual Casimir pressure. But for larger fields, e.g. for $B \sim 10^5$ G it becomes larger; one may obtain then pressures up to $P_{0e\perp} \sim 10^{-9} dyn\ cm^{-2}$. For a distance between plates $d = 0.1cm$, it gives $P_{0C} \sim 10^{-14} dyn\ cm^{-2}$, (see below) i.e., five orders of magnitude smaller than $P_{0e\perp}$. Our results show that quantum vacuum in a constant magnetic field may exert pressures, either positive or negative, which means *a transfer of momentum from vacuum to real particles or macroscopic bodies* (as well as in Casimir effect, see below). A similar idea is approached from classical grounds in [12].

ASTROPHYSICAL AND COSMOLOGICAL CONSEQUENCES?

Quantum vacuum energy has been suggested as a possible candidate to dark energy, leading to a repulsive gravity, equivalent to a cosmological constant [13],[14]. The condition $3p + \rho < 0$ is expected to be fulfilled in Einstein equations assuming the energy density $\rho > 0$, and as a consequence the average pressure $p < 0$, which means $w = (p/\rho) < -1/3$.

In the classical isotropic case it is $\rho + 3p = \mathscr{T}_{\mu\mu}$. That is why it is of special interest to consider the trace of the tensor $\mathscr{T}_{\mu\mu}$ in both studied cases. In the Casimir case one has $\mathscr{T}_{\mu\mu}^{C} = 2\Omega_C < 0$ and in presence of the magnetic field it is $\mathscr{T}_{\mu\mu}^{0e} = -2\Omega_{0e} - 2\mathscr{M}B < 0$[4].

[4] Note that the trace $\mathscr{T}_{\mu}^{0e\mu} = -4\Omega_{0e} - 2B\mathscr{M}_{0e}$ whereas $\mathscr{T}_{\mu}^{C\mu} = 0$ ($\mathscr{T}_4^4 = -\Omega$ in both cases)

It is easy to check that in the magnetic field and Casimir cases both the average pressure $\mathscr{T}_{ii}/3$ and the energy density, are negative, since it is $\langle p_{0e}\rangle = -\Omega_{0e} + 2B\mathscr{M}/3 < 0$, but $\langle p_{0e}\rangle/\Omega_{0e} > 1$, and for the Casimir case we obtain similarly $\langle p_{0C}\rangle = \Omega_{0C}/3 < 0$, (and $\langle p_{0C}\rangle/\Omega_{0C} = 1/3$ as in blackbody radiation). It is easy to check that for the magnetic field case, the expression for the energy-momentum tensor of vacuum is Lorentz-invariant with regard to inertial frames moving parallel to B. It is also easy to check that the Casimir energy-momentum tensor defined by (4), (6) and (7) remains invariant with regard to Lorentz transformations to inertial frames parallel to the plates.

In considering the effect of magnetized vacuum, we shall assume the magnetic lines of force in intergalactic space as describing curves in all directions (there is no preferred direction for the magnetic field B) , or either, we may assume that there are magnetic domains randomly distributed in space, assuming in each of them the magnetic field as constant, so as to give an isotropic spatial average of the energy-momentum tensor. Concerning the energy term, it must contain the contribution of the average density of the matter creating the magnetic field.

For superdense magnetized matter the pressure transverse to B is $P_\perp = -\Omega - B\mathscr{M}$ [11]. When $-\Omega \leq B\mathscr{M}$, the transverse pressure vanishes or becomes negative, leading to unstable conditions: the gravitational pressure exerted by the body cannot be balanced by matter pressure, the outcome being an anisotropic collapse [8]. For the energy density under these conditions one can write $U \leq N\mu - B\mathscr{M}$. We observe that magnetic fields decrease the energy density, although we expect that $U \geq 0$ in any case, i.e. under stable conditions the (negative) magnetic energy never exceeds in modulus the rest energy. (This happens for a gas of charged vector bosons, whose ground state has a decreasing but non-vanishing effective mass $\sqrt{M^2c^4 - eBc\hbar} > 0$ [15] (for increasing B)). Under that assumption, the total energy density would be $U \leq N\mu - B\mathscr{M} + \Omega_{0e} > 0$. Thus, we expect that even if the average magnetic vacuum pressure taken in large regions of space has a negative sign, the average energy density of magnetic field+sources should be positive. By comparing (5) with the estimated density of visible matter (around 10^{-10}erg/cm^3) [16], it would mean a field of 10^5G. For present cosmology this is too large, since the estimates for the intergalactic magnetic fields are in the range $10^{-6} - 10^{-9}$ G [17]. But the mechanism is interesting in the early universe, since if we assume the existence of some regions of space having a distribution of strongly magnetized matter, it might be possible to have vacuum average negative pressures leading inside these regions to $\rho + 3\langle p\rangle < 0$, with $\rho > 0$.

The main result of the previous discussion is that QED vacuum under the action of external fields having axial symmetry, provide the way of getting finite vacuum negative pressures and negative average energies.

ACKNOWLEDGMENTS

H.P.R. is especially indebted to G. Altarelli, M. Chaichian, C. Montonen, D. Schwartz and R. Sussmann for enlightening comments. Both authors thank H. Mosquera Cuesta and A.E. Shabad for discussions.

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