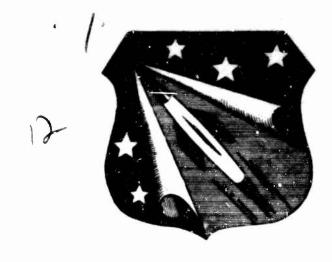
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# OPTIMUM STAGE-WEIGHT DISTRIBUTION OF MULTISTAGE ROCKETS

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#### ABSTRACT

In this analysis a generalized method is developed for determining the optimum stage-weight distribution for multistage rockets. Inclusion of the variations in structural factors with stage weights in the optimization process is shown to lead to a more generalized set of optimum conditions. Expression of all rocket weight parameters in terms of the stage weights allows for convenient optimization as well as for a comparison with previous optimization methods.

This approach permits improved optimum design over existing methods for maximizing payload ratios for given ranges and for maximizing ranges for given payload ratios. An evaluation of previous methods is included for comparison purposes, and the limitations of these previous methods are discussed.

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#### I. INTRODUCTION

In recent years, the optimization of multistaged rockets has received considerable attention. Since the performance of missiles and space vehicles is sensitive to small changes in design, optimization procedures are of great importance. In References 1 through 11 methods were developed for determining "optimum" stage mass ratio distributions. None of these methods, however, allowed for variations of structural factor with stage weight. Consequently, the referenced methods do not yield truly optimum stage mass ratio distributions.

The purpose of this paper is first, to derive, in terms of the stage weights, a more general design optimization that allows for variations of structural factors with stage weights, and second, to evaluate the limitations of previous design criteria.



# NOMENCLATURE

P	Payload ratio, as defined by Equation (1)
$\mathbf{w}_{\mathbf{L}}$	Payload weight
$\mathbf{w_o^{(1)}}$	Gross vehicle weight
v <sub>bo</sub>	Burnout velocity
I <sub>j</sub>	Specific impulse of the jth stage
g	Acceleration of gravity
r <sub>j</sub>	Mass ratio of jth stage
R	Range
D and B	Empirical parameters for range Equation (3)
•	Lumped velocity requirement term
$\sigma_{\mathbf{i}}$	Structural factor of ith stage
$\mathbf{w_i}$	Weight of the i <sup>th</sup> stage
W <sub>pi</sub>	Propellant weight of the ith stage
$\mathbf{w}_{o}^{(i)}$	Gross weight of ith stage as defined by equation (11)
N	Total number of stages
∂w <sub>i</sub> σ <sub>i</sub>	Partial derivative of $\phi$ with respect to $W_i$ , keeping $\sigma_i$ fixed
$\left  \frac{\partial \phi}{\partial \sigma_i} \right  \mathbf{w_i}$	Partial derivative of φ with respect to σ <sub>i</sub> , keeping W <sub>i</sub> fixed
W <sub>bo</sub> (i)	Burnout weight of the ith stage, as defined by Equations (12) and (13)

#### II. OPTIMIZATION PARAMETERS

## A. Performance Parameters

The performance capability of a multistage rocket vehicle can be described by two equations.

$$P = \frac{W^{(1)}}{W_L} \tag{1}$$

$$V_{bo} = \sum_{j=1}^{N} I_{j} g \ln r_{j} - \delta V$$
 (2)

where  $\delta V$  represents the velocity losses associated with gravity and drag. The drag losses are primarily dependent upon the initial thrust-to-weight ratio,  $N_o$ , and on the quantity,  $W_o^{(1)}/C_DA$ .

Equation (2) can be rewritten in terms of range, R, for a ballistic missile: (Reference 12)

$$R = D\left[\prod_{i=1}^{N} r_i^{I_i/B} - 1\right]$$
 (3)

where B is very insensitive to changes in  $N_0$  and  $W_0^{(1)}/C_DA$ , while the parameter D is fairly sensitive to such changes.

Let

$$\phi = \prod_{j=1}^{N} r_{j}^{Ij}$$
 (4)

and the theoretical velocity,

4

$$V_{t} = V_{bo} + \delta V \tag{5}$$

Then from Eqs. (2), (3), and (4),

$$\phi = \prod_{j=1}^{N} x_j^{ij} = e^{\frac{V_t}{g}}$$
 (6)

OT

$$\phi = \prod_{j=1}^{N} r_j^{I_j} = \left(\frac{R+D}{D}\right)^B$$
 (7)

Thus, for a given initial thrust-to-weight ratio and mission, the velocity requirements may be lumped into a fixed single term,  $\phi$ . The range equation already provides for velocity losses in the empirical constants D and B used for any particular configuration.

When there are at least two stages and when all the specific impulses are known. P and \$\phi\$ do not uniquely define the mass ratios of the various stages. Consequently, proper selection (optimization) of mass ratios for either maximum payload at a given range or maximum range at a given payload ratio is required.

#### B. Structural Factor Parameters

The structural factor,  $\sigma_i$ , for the i<sup>th</sup> stage is given by:

$$\sigma_{i} = \frac{W_{i} - W_{p_{i}}}{W_{i}} \tag{8}$$

Expressed in terms of the weight of the ith stage, the following scaling laws are assumed to hold:

$$\sigma_{i} = C_{i} W_{i}^{n_{i}-1}$$
 (9)

where C<sub>i</sub> and n<sub>i</sub> are empirical constants for each stage subject to the selection of propellant feed systems, auxiliary systems, etc.



## C. Use of Stage-Weight Parameters

The mass ratio of the ith stage can be defined by:

$$r_i = \frac{W_o^{(i)}}{W_{bo}^{(i)}} = \frac{W_o^{(i)}}{W_o^{(i)} - W_{bi}}$$
 (10)

where

$$W_o^{(i)} = \sum_{j=i}^{N} W_j + W_L$$
 (11)

and

$$W_{bo}^{(i)} = W_{o}^{(i)} - W_{pi}$$
 (12)

From Eqs. (8), (11), and (12), the following useful relations can be derived for an N-stage rocket:

 $W_{bo}^{(N)} = \sigma_N W_N + W_L = \sigma_N W_o^{(N)} + (1 - \sigma_N) W_L$ 

$$W_{bo}^{(1)} = \sigma_{1} W_{1} + W_{o}^{(2)} = \sigma_{1} W_{o}^{(1)} + (1 - \sigma_{1}) W_{o}^{(2)}$$

$$W_{bo}^{(2)} = \sigma_{2} W_{2} + W_{o}^{(3)} = \sigma_{2} W_{o}^{(2)} + (1 - \sigma_{2}) W_{o}^{(3)}$$

$$(13)$$

Combining Eqs. (9), (10), (11) and (13),

$$r_{i} = \frac{\sum_{j=i}^{N} w_{j} + w_{L}}{C_{i} W_{i}^{n_{i}} + \sum_{j=i+1}^{N} W_{j} + W_{L}}$$
(14)

Substituting (11) into (1),

$$P = \frac{\sum_{j=1}^{N} W_j + W_L}{W_L}$$
 (15)

where the payload ratio is now expressed in terms of the stage weights. By substituting (14) into (6) or (7),  $\phi$  can also be expressed in terms of the stage weights:

$$\phi = \prod_{k=1}^{N} \left( \frac{\sum_{j=k}^{N} w_{j} + w_{L}}{C_{k} w_{k}^{n_{k}} + \sum_{j=k+1}^{N} w_{j} + w_{L}} \right)^{I_{k}}$$
(16)

#### III. THE GENERAL OPTIMIZATION

#### A. Lagrangian Multiplier Technique

A necessary condition that a function  $f(x_1, x_2, ..., x_N)$  of N variables  $x_1, x_2, ..., x_N$  have a stationary value is that

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_N} dx_N = 0$$
 (17)

for all permissible values of the differentials  $dx_1, dx_2, \ldots, dx_N$ . If, however, the N variables are not independent, but are related by another condition of the form  $\psi(x_1, x_2, \ldots, x_N) = 0$ , then the procedure of introducing the so-called Lagrange multiplier may be conveniently employed:

Now if  $\lambda$  is so chosen that

$$\frac{\partial f}{\partial x_1} + \lambda \frac{\partial \Psi}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} + \lambda \frac{\partial \Psi}{\partial x_2} = 0$$

$$\frac{\partial f}{\partial x_N} + \lambda \frac{\partial \Psi}{\partial x_N} = 0$$

$$\psi(x_1, x_2, \dots, x_N) = 0$$
(18)

then the necessary condition for an extremum of  $f(x_1, x_2, ..., x_N)$  will be satisfied. The quantity  $\lambda$  is known as a Lagrangian multiplier.

# B. Application of the Lagrangian Multiplier Technique to Optimum Rocket Design

The Lagrangian multiplier method may be used to optimize P subject to a fixed  $\phi$  or vice versa. In fact the partial differential equations in (18) would be the same for either optimization; only the constraining equation would be different.

In general, the conditions (18) applied to optimum stage weights become:

$$\frac{\partial P}{\partial W_1} + \lambda \frac{\partial \phi}{\partial W_2} = 0$$

$$\frac{\partial P}{\partial W_2} + \lambda \frac{\partial \phi}{\partial W_2} = 0$$

$$(19)$$

$$\frac{\partial P}{\partial W_N} + \lambda \frac{\partial \phi}{\partial W_N} = 0$$

and either  $\phi$  or P = constant

## C. Criticism of Previous Methods

The optimization conditions expressed by (19) guarantee a minimum P for constant  $\phi$  or vice versa, since variation of structural factor with stage weight is included in the optimization process. In previous methods (References 1 to 11), this variation was not included. To evaluate these methods,  $\phi$  can be re-expressed as:

$$\phi = \phi(W_1, W_2, ..., W_{N'}, ..., \sigma_1, \sigma_2, ..., \sigma_N)$$
 (20)

Then:

$$\frac{\partial \phi}{\partial W_{1}} = \frac{\partial \phi}{\partial W_{1}} \Big|_{\sigma_{1}} + \frac{\partial \phi}{\partial \sigma_{1}} \Big|_{W_{1}} \frac{d\sigma_{1}}{dW_{1}}$$

$$\frac{\partial \phi}{\partial W_{2}} = \frac{\partial \phi}{\partial W_{2}} \Big|_{\sigma_{2}} + \frac{\partial \sigma}{\partial \sigma_{2}} \Big|_{W_{2}} \frac{d\sigma_{2}}{dW_{2}}$$

$$\frac{\partial \phi}{\partial W_{N}} = \frac{\partial \phi}{\partial W_{N}} \Big|_{\sigma_{N}} + \frac{\partial \phi}{\partial \sigma_{N}} \Big|_{W_{N}} \frac{d\sigma_{N}}{dW_{N}}$$
(21)

Equations (19) then become

$$\frac{\partial P}{\partial W_{1}} + \lambda \frac{\partial \phi}{\partial W_{1}} \Big|_{\sigma_{1}} + \lambda \frac{\partial \phi}{\partial \sigma_{1}} \Big|_{W_{1}} \frac{d\sigma_{1}}{dW_{1}} = 0$$

$$\frac{\partial P}{\partial W_{2}} + \lambda \frac{\partial \phi}{\partial W_{2}} \Big|_{\sigma_{2}} + \lambda \frac{\partial \phi}{\partial \sigma_{2}} \Big|_{W_{2}} \frac{d\sigma_{2}}{dW_{2}} = 0$$

$$\frac{\partial P}{\partial W_{N}} + \lambda \frac{\partial \phi}{\partial W_{N}} \Big|_{\sigma_{N}} + \lambda \frac{\partial \phi}{\partial \sigma_{N}} \Big|_{W_{N}} \frac{d\sigma_{N}}{dW_{N}} = 0$$
(22)

and either P or \( \phi \) constant.

When no variation of structural factor with stage weights is considered, the corresponding expressions for "optimization" can easily be shown to be

See Appendix

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$$\frac{\partial P}{\partial W_1} + \lambda \frac{\partial \phi}{\partial W_1} \bigg|_{\sigma_1} = 0$$

$$\frac{\partial P}{\partial W_2} + \lambda \frac{\partial \phi}{\partial W_2} \bigg|_{\sigma_2} = 0$$

(23,

$$\frac{\partial P}{\partial W_N} + \lambda \frac{\partial \phi}{\partial W_N} \bigg|_{\sigma_N} = 0$$

and either P or  $\phi$  = constant.

Since relations (22) are the true optimization conditions then, in order for relations (23) to be optimum, (22) and (23) must be compatible. It is clear that, in order for (22) and (23) to be compatible, the relations

$$\frac{\partial \phi}{\partial \sigma_1}\Big|_{W_1} \frac{d\sigma_1}{dW_1} = 0$$

$$\frac{\partial \phi}{\partial \sigma_2} \bigg|_{W_2} \frac{d\sigma_2}{dW_2} = 0 \tag{24}$$

(24)

$$\left. \frac{\partial \phi}{\partial \sigma_{\mathbf{N}}} \right|_{\mathbf{W}_{\mathbf{N}}} \frac{\mathrm{d} \sigma_{\mathbf{N}}}{\mathrm{d} W_{\mathbf{N}}} = 0$$

must hold.

Since (24) cannot be valid, except in the nonrealistic case where the structural factor does not vary with stage weight, Eqs. (22) and (23) are not compatible. Hence, Eqs. (23) do not represent a true optimization criterion for realistic rocket design. The actual discrepancy in the design criterion in using (23) rather than (22) will depend upon the relative magnitudes of the terms in (22) that have the coefficient  $\lambda$ . If the second term of coefficient  $\lambda$  in (22) is negligible compared to the first term of coefficient  $\lambda$ , then (23) represents a realistic optimization. In general, however, this is not the case.

#### D. Reduction of the Optimization Equations to Simpler Form

From Eq. (15), for fixed  $W_L$ :

$$\frac{\partial P}{\partial W_1} = \frac{\partial P}{\partial W_2} = \dots = \frac{\partial P}{\partial W_N} = \frac{1}{W_L}$$
 (25)

Substituting (25) into (19) and eliminating common terms, the optimization equations reduce to:

$$\frac{\partial \phi}{\partial W_1} = \frac{\partial \phi}{\partial W_2} = --- = \frac{\partial \phi}{\partial W_N}$$
 (26)

and either  $\phi$  or P = constant

#### IV. APPLICATION TO OPTIMUM THREE-STAGE ROCKET DESIGN

For a three-stage rocket, Eqs. (15) and (16) become

$$P = \frac{W_1 + W_2 + W_3 + W_L}{W_L}$$
 (27)

and

$$\phi = \left(\frac{w_1 + w_2 + w_3 + w_L}{C_1 w_1^{n_1} + w_2 + w_3 + w_L}\right)^{I_1} \left(\frac{w_2 + w_3 + w_L}{C_2 w_2^{n_2} + w_3 + w_L}\right)^{I_2} \left(\frac{w_3 + w_L}{C_3 w_3^{n_3} + w_L}\right)^{I_3}$$
(28)

Accordingly,

$$\frac{\partial \phi}{\partial W_{1}} = \frac{\phi I_{1} \left[ (1 - n_{1}) C_{1} W_{1}^{n_{1}} + (1 - n_{1} C_{1} W_{1}^{n_{1}-1}) (W_{2} + W_{3} + W_{L}) \right]}{\left( W_{1} + W_{2} + W_{3} + W_{L} \right) \left( C_{1} W_{1}^{n_{1}} + W_{2} + W_{3} + W_{L} \right)}$$
(29)

$$\frac{\partial \phi}{\partial W_{2}} = \frac{\phi I_{1} \left( C_{1} W_{1}^{n_{1}-1} - 1 \right) W_{1}}{\left( W_{1} + W_{2} + W_{3} + W_{L} \right) \left( C_{1} W_{1}^{n_{1}} + W_{2} W_{3} + W_{L} \right)} + \frac{\phi I_{2} \left[ (1 - n_{2}) C_{2} W_{2}^{n_{2}} + \left( 1 - n_{2} C_{2} W_{2}^{n_{2}-1} \right) (W_{3} + W_{L}) \right]}{\left( W_{2} + W_{3} + W_{L} \right) \left( C_{2} W_{2}^{n_{2}} + W_{3} + W_{L} \right)} \tag{30}$$

and

)

$$\frac{\partial \phi}{\partial W_{3}} = \frac{\phi I_{1} \left(C_{1} W_{1}^{n_{1}-1} - 1\right) W_{1}}{\left(W_{1} + W_{2} + W_{3} + W_{L}\right) \left(C_{1} W_{1}^{n_{1}} + W_{2} + W_{3} + W_{L}\right)} + \frac{\phi I_{2} \left(C_{2} W_{2}^{n_{2}-1} - 1\right) W_{2}}{\left(W_{2} + W_{3} + W_{L}\right) \left(C_{2} W_{2}^{n_{2}} + W_{3} + W_{L}\right)} + \frac{\phi I_{3} \left[\left(1 - n_{3}\right)C_{3} W_{3}^{n_{3}} + \left(1 - n_{3}C_{3} W_{3}^{n_{3}-1}\right) W_{L}\right]}{\left(W_{3} + W_{L}\right) \left(C_{3} W_{3}^{n_{3}} + W_{L}\right)} \tag{31}$$

Substituting (29), (30), and (31) into the optimization equations,

$$\frac{\partial \phi}{\partial W_1} = \frac{\partial \phi}{\partial W_2} = \frac{\partial \phi}{\partial W_3}.$$

and factoring out \( \phi \) yields two independent nonlinear equations:

$$\frac{I_{1}\left(1-n_{1}C_{1}W_{1}^{n_{1}-1}\right)}{\left(1-C_{1}W_{1}^{n_{1}-1}\right)}\left[1-C_{1}W_{1}^{n_{1}-1}\left(\frac{W_{1}+W_{2}+W_{3}+W_{L}}{C_{1}W_{1}^{n_{1}}+W_{2}+W_{3}+W_{L}}\right)\right]$$

$$= I_{2}\left[1-n_{2}C_{2}W_{2}^{n_{2}-1}\left(\frac{W_{2}+W_{3}+W_{L}}{C_{2}W_{2}^{n_{2}}+W_{3}+W_{L}}\right)\right] \tag{32}$$



$$I_{2} = \frac{\left(1 - n_{2}C_{2}W_{2}^{n_{2}-1}\right)}{\left(1 - C_{2}W_{2}^{n_{2}-1}\right)} \left[1 - C_{2}W_{2}^{n_{2}-1} \left(\frac{W_{2} + W_{3} + W_{L}}{C_{2}W_{2}^{n_{2}} + W_{3} + W_{L}}\right)\right]$$

$$= I_{3} \left[1 - n_{3}C_{3}W_{3}^{n_{3}-1} \left(\frac{W_{3} + W_{L}}{C_{3}W_{3}^{n_{3}} + W_{L}}\right)\right]$$
(33)

Equations (32) and (33) together with the constraint equation,  $\phi$  or P = constant, constitute three equations in terms of the three unknowns  $W_1$ ,  $W_2$ , and  $W_3$ . For specified sets of values of  $I_1$ ,  $I_2$ ,  $I_3$ ,  $C_1$ ,  $C_2$ ,  $C_3$ ,  $n_1$ ,  $n_2$ ,  $n_3$ , and  $\phi$  or P (depending upon which type of optimization is desired), the three equations can be solved by iteration techniques to yield optimum values of  $W_1$ , and  $W_2$ , and  $W_3$ . By employing Equations (11) and (13), the other rocket parameters can also be determined.

#### V. APPLICATION TO OPTIMIZING TWO-STAGE ROCKET DESIGN

Analogous to the three stage optimization, the optimization conditions for a two stage rocket become:

$$\frac{\partial \phi}{\partial W_1} = \frac{\partial \phi}{\partial W_2}$$
and  $\phi$  or  $P = constant$  (34)

which yield:

$$I_{1} \frac{\left(1 - n_{1} C_{1} W_{1}^{n_{1} - 1}\right)}{\left(1 - C_{1} W_{1}^{n_{1} - 1}\right)} \left[1 - C_{1} W_{1}^{n_{1} - 1} \left(\frac{W_{1} + W_{2} + W_{L}}{C_{1} W_{1}^{n_{1} + W_{2} + W_{L}}}\right)\right]$$

$$= I_{2} \left[1 - n_{2} C_{2} W_{2}^{n_{2} - 1} \left(\frac{W_{2} + W_{L}}{C_{2} W_{2}^{n_{2} + W_{L}}}\right)\right]$$
(35)

Equation (35) together with the contraint,  $\phi$  or P = constant, can be solved by iteration to yield optimal values of  $W_1$  and  $W_2$ .

# VI. APPLICATION TO N-STAGE ROCKET OPTIMIZATION

If iterative techniques can be expanded to include solving N simultaneous nonlinear equations, then the process described above can be extended to any number of stages. In the case of a large number of stages, computerized random search techniques might be employed to solve for optimum values of the stage weights. Once the optimum stage weights are obtained, these values can be substituted into Eqs. (11) and (13) to solve for the other rocket parameters.

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#### APPENDIX

Further comparison with previous methods writing  $\phi$  for a three stage recket in terms of  $\pi_1$ ,  $\pi_2$ , and  $\pi_3$ .

$$\phi = \left(\frac{W_1 + W_2 + W_3 + W_L}{\sigma_1 W_1 + W_2 + W_3 + W_L}\right)^{1} \left(\frac{W_2 + W_3 + W_L}{\sigma_2 W_2 + W_3 + W_L}\right)^{1} \left(\frac{W_3 + W_L}{\sigma_3 W_3 + W_L}\right)^{1}$$
(36)

Then

$$\frac{\partial \phi}{\partial W_1} = \frac{I_1 \phi (1 - \sigma_1) (W_2 + W_3 + W_L)}{(\sigma_1 W_1 + W_2 + W_3 + W_L) (W_1 + W_2 + W_3 + W_L)}$$
(37)

$$\frac{\partial \phi}{\partial W_{2}} = \frac{\frac{(\sigma_{1}W_{1} + W_{2} + W_{3} + W_{L})(W_{1} + W_{2} + W_{3} + W_{L})}{(\sigma_{1}W_{1} + W_{2} + W_{3} + W_{L})(W_{1} + W_{2} + W_{3} + W_{L})} + \frac{I_{2}\phi(1 - \sigma_{2})(W_{3} + W_{L})}{(\sigma_{2}W_{2} + W_{3} + W_{L})(W_{2} + W_{3} + W_{L})}$$
(38)

and

$$\begin{array}{c|c}
\hline
8\phi \\
8W_{3} \\
\sigma_{3}
\end{array} = (\overline{\sigma_{1}W_{1} + W_{2} + W_{3} + W_{L}})(W_{1} + W_{2} + W_{3} + W_{L})$$

$$+ \frac{I_{2}\phi(\sigma_{2} - 1)W_{2}}{(\sigma_{2}W_{2} + W_{3} + W_{L})(W_{2} + W_{3} + W_{L})}$$

$$+ \frac{I_{3}\phi(1 - \sigma_{3})W_{L}}{(\sigma_{3}W_{3} + W_{L})(W_{3} + W_{L})}$$
(39)

Substuting (37), (38) and (39) into the conditions from (23):

$$\frac{\partial \phi}{\partial W_1} \Big|_{\sigma_1} = \frac{\partial \phi}{\partial W_2} \Big|_{\sigma_2} = \frac{\partial \phi}{\partial W_3} \Big|_{\sigma_3}$$

result in:

$$I_{1} \left[ 1 - \sigma_{1} \left( \frac{W_{1} + W_{2} + W_{3} + W_{L}}{\sigma_{1} W_{1} + W_{2} + W_{3} + W_{L}} \right) \right] = I_{2} \left[ 1 - \sigma_{2} \left( \frac{W_{2} + W_{3} + W_{L}}{\sigma_{2} W_{2} + W_{3} + W_{L}} \right) \right]$$

$$= I_{3} \left[ 1 - \sigma_{3} \left( \frac{W_{3} + W_{L}}{\sigma_{3} W_{3} + W_{L}} \right) \right]$$

$$(40)$$

Equation (40) represents the relations corresponding to previous methods (1-11) for design criteria. By using Eqs. (11) and (13), then Eqs. (40), (32), and (33) can be written for comparison purposes in terms of the mass ratios. Equation (40) then becomes

$$I_1(1 - \sigma_1 r_1) = I_2(1 - \sigma_2 r_2) = I_3(1 - \sigma_3 r_3)$$
 (41)

which is the more familiar form usually seen in the references listed above. Equations (32) and (33) become:

$$I_{1} = \frac{(1 - n_{1} \sigma_{1})}{(1 - \sigma_{1})} = I_{2}(1 - n_{2}\sigma_{2}r_{2})$$
 (42)

$$I_2 = \frac{(1 - n_2 \sigma_2)}{(1 - \sigma_2)} (1 - \sigma_2 r_2) = I_3 (1 - n_3 \sigma_3 r_3)$$
 (43)

or combining (42) and (43):

$$I_{1} = \frac{(1 - n_{1} \sigma_{1})}{(1 - \sigma_{1})} (1 - \sigma_{1} r_{1}) = I_{2}(1 - n_{2} \sigma_{2} r_{2}) = I_{2}(1 - n_{2}) + n_{2} \frac{I_{3}(1 - \sigma_{2})}{(1 - n_{2} \sigma_{2})} \times (1 - n_{3} \sigma_{3} r_{3})$$

$$(1 - n_{3} \sigma_{3} r_{3})$$

$$(44)$$

which represent more inclusive optimization conditions. Comparison of (41) with (44) indicates the significant difference between previous "optimisation" and the more realistic optimization criteria.