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OPTIMUM STAGE-WEIGHT DISTRIBUTION OF MULTISTAGE ROCKETS

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ABSTRACT

In this analysis a generalized method is developed for determining the optimum stage-weight distribution for multistage rockets. Inclusion of the variations in structural factors with stage weights in the optimization process is shown to lead to a more generalized set of optimum conditions. Expression of all rocket weight parameters in terms of the stage weights allows for convenient optimization as well as for a comparison with previous optimization methods.

This approach permits improved optimum design over existing methods for maximizing payload ratios for given ranges and for maximizing ranges for given payload ratios. An evaluation of previous methods is included for comparison purposes, and the limitations of these previous methods are discussed.

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CONTENTS

I. INTRODUCTION

In recent years, the optimization of multistaged rockets has received considerable attention. Since the performance of missiles and space vehicles is sensitive to small changes in design, optimization procedures are of great importance. In References 1 through 11 methods were developed for determining "optimum" stage mass ratio distributions. None of these methods, however, allowed for variations of structural factor with stage weight. Consequently, the referenced methods do not yield truly optimum stage mass ratio distributions.

The purpose of this paper is first, to derive, in terms of the stage weights, a more general design optimization that allows for variations of structural factors with stage weights, and second, to evaluate the limitations of previous design criteria.

NOMENCLATURE

P	Payload ratio, as defined by Equation (1)
W_L	Payload weight
$W_o^{(1)}$	Gross vehicle weight
V_{bo}	Burnout velocity
I_j	Specific impulse of the j^{th} stage
g	Acceleration of gravity
r_j	Mass ratio of j^{th} stage
R	Range
D and B	Empirical parameters for range Equation (3)
ϕ	Lumped velocity requirement term
σ_i	Structural factor of i^{th} stage
W_i	Weight of the i^{th} stage
W_{pi}	Propellant weight of the i^{th} stage
$W_o^{(i)}$	Gross weight of i^{th} stage as defined by equation (11)
N	Total number of stages
$\left.\frac{\partial \phi}{\partial W_i}\right\|_{\sigma_i}$	Partial derivative of ϕ with respect to W_i, keeping σ_i fixed
$\left.\frac{\partial \phi}{\partial \sigma_i}\right\|_{W_i}$	Partial derivative of ϕ with respect to σ_i, keeping W_i fixed
$W_{bo}(i)$	Burnout weight of the i^{th} stage, as defined by Equations (12) and (13)

II. OPTIMIZATION PARAMETERS

A. Performance Parameters

The performance capability of a multistage rocket vehicle can be described by two equations.

$$P = \frac{W_o^{(1)}}{W_L} \tag{1}$$

$$V_{bo} = \sum_{j=1}^{N} I_j \, g \ln r_j - \delta V \tag{2}$$

where δV represents the velocity losses associated with gravity and drag. The drag losses are primarily dependent upon the initial thrust-to-weight ratio, N_o, and on the quantity, $W_o^{(1)}/C_D A$.

Equation (2) can be rewritten in terms of range, R, for a ballistic missile: (Reference 12)

$$R = D\left[\prod_{i=1}^{N} r_i^{I_i/B} - 1\right] \tag{3}$$

where B is very insensitive to changes in N_o and $W_o^{(1)}/C_D A$, while the parameter D is fairly sensitive to such changes.

Let

$$\phi = \prod_{j=1}^{N} r_j^{I_j} \tag{4}$$

and the theoretical velocity,

$$V_t = V_{bo} + \delta V \tag{5}$$

Then from Eqs. (2), (3), and (4),

$$\phi = \prod_{j=1}^{N} r_j^{I_j} = e^{\frac{V_t}{g}} \tag{6}$$

or

$$\phi = \prod_{j=1}^{N} r_j^{I_j} = \left(\frac{R + D}{D}\right)^B \tag{7}$$

Thus, for a given initial thrust-to-weight ratio and mission, the velocity requirements may be lumped into a fixed single term, ϕ. The range equation already provides for velocity losses in the empirical constants D and B used for any particular configuration.

When there are at least two stages and when all the specific impulses are known, P and ϕ do not uniquely define the mass ratios of the various stages. Consequently, proper selection (optimization) of mass ratios for either maximum payload at a given range or maximum range at a given payload ratio is required.

B. Structural Factor Parameters

The structural factor, σ_i, for the i^{th} stage is given by:

$$\sigma_i = \frac{W_i - W_{p_i}}{W_i} \tag{8}$$

Expressed in terms of the weight of the i^{th} stage, the following scaling laws are assumed to hold:

$$\sigma_i = C_i W_i^{n_i - 1} \tag{9}$$

where C_i and n_i are empirical constants for each stage subject to the selection of propellant feed systems, auxiliary systems, etc.

C. Use of Stage-Weight Parameters

The mass ratio of the i^{th} stage can be defined by:

$$r_i = \frac{W_o^{(i)}}{W_{bo}^{(i)}} = \frac{W_o^{(i)}}{W_o^{(i)} - W_{pi}} \tag{10}$$

where

$$W_o^{(i)} = \sum_{j=i}^{N} W_j + W_L \tag{11}$$

and

$$W_{bo}^{(i)} = W_o^{(i)} - W_{pi} \tag{12}$$

From Eqs. (8), (11), and (12), the following useful relations can be derived for an N-stage rocket:

$$\begin{aligned} W_{bo}^{(1)} &= \sigma_1 W_1 + W_o^{(2)} = \sigma_1 W_o^{(1)} + (1 - \sigma_1) W_o^{(2)} \\ W_{bo}^{(2)} &= \sigma_2 W_2 + W_o^{(3)} = \sigma_2 W_o^{(2)} + (1 - \sigma_2) W_o^{(3)} \\ &\text{---} \\ &\text{---} \\ W_{bo}^{(N)} &= \sigma_N W_N + W_L = \sigma_N W_o^{(N)} + (1 - \sigma_N) W_L \end{aligned} \tag{13}$$

Combining Eqs. (9), (10), (11) and (13),

$$r_i = \frac{\sum_{j=i}^{N} W_j + W_L}{C_i W_i^{n_i} + \sum_{j=i+1}^{N} W_j + W_L} \tag{14}$$

Substituting (11) into (1),

$$P = \frac{\sum_{j=1}^{N} W_j + W_L}{W_L} \tag{15}$$

where the payload ratio is now expressed in terms of the stage weights. By substituting (14) into (6) or (7), ϕ can also be expressed in terms of the stage weights:

$$\phi = \prod_{k=1}^{N} \left(\frac{\sum_{j=k}^{N} W_j + W_L}{C_k W_k^{n_k} + \sum_{j=k+1}^{N} W_j + W_L} \right)^{I_k} \tag{16}$$

III. THE GENERAL OPTIMIZATION

A. Lagrangian Multiplier Technique

A necessary condition that a function $f(x_1, x_2, \ldots, x_N)$ of N variables $x_1, x_2, \ldots, x_N$ have a stationary value is that

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_N} dx_N = 0 \tag{17}$$

for all permissible values of the differentials $dx_1, dx_2, \ldots, dx_N$. If, however, the N variables are not independent, but are related by another condition of the form $\psi(x_1, x_2, \ldots, x_N) = 0$, then the procedure of introducing the so-called Lagrange multiplier may be conveniently employed:
Now if λ is so chosen that

$$\begin{aligned}
\frac{\partial f}{\partial x_1} + \lambda \frac{\partial \Psi}{\partial x_1} &= 0 \\
\frac{\partial f}{\partial x_2} + \lambda \frac{\partial \Psi}{\partial x_2} &= 0 \\
&\cdots\cdots \\
\frac{\partial f}{\partial x_N} + \lambda \frac{\partial \Psi}{\partial x_N} &= 0 \\
\psi(x_1, x_2, \ldots\ldots, x_N) &= 0
\end{aligned} \tag{18}$$

then the necessary condition for an extremum of $f(x_1, x_2, \ldots, x_N)$ will be satisfied. The quantity λ is known as a Lagrangian multiplier.

B. Application of the Lagrangian Multiplier Technique to Optimum Rocket Design

The Lagrangian multiplier method may be used to optimize P subject to a fixed ϕ or vice versa. In fact the partial differential equations in (18) would be the same for either optimization; only the constraining equation would be different.

In general, the conditions (18) applied to optimum stage weights become:

$$\begin{aligned} \frac{\partial P}{\partial W_1} + \lambda \frac{\partial \phi}{\partial W_1} &= 0 \\ \frac{\partial P}{\partial W_2} + \lambda \frac{\partial \phi}{\partial W_2} &= 0 \\ &\text{----------} \\ &\text{----------} \\ \frac{\partial P}{\partial W_N} + \lambda \frac{\partial \phi}{\partial W_N} &= 0 \end{aligned} \tag{19}$$

and either ϕ or P = constant

C. Criticism of Previous Methods

The optimization conditions expressed by (19) guarantee a minimum P for constant ϕ or vice versa, since variation of structural factor with stage weight is included in the optimization process. In previous methods (References 1 to 11), this variation was not included. To evaluate these methods, ϕ can be re-expressed as:

$$\phi = \phi(W_1, W_2, \ldots, W_N, \ldots, \sigma_1, \sigma_2, \ldots, \sigma_N) \tag{20}$$

Then:

$$\frac{\partial\phi}{\partial W_1} = \left.\frac{\partial\phi}{\partial W_1}\right|_{\sigma_1} + \left.\frac{\partial\phi}{\partial\sigma_1}\right|_{W_1} \frac{d\sigma_1}{dW_1}$$

$$\frac{\partial\phi}{\partial W_2} = \left.\frac{\partial\phi}{\partial W_2}\right|_{\sigma_2} + \left.\frac{\partial\phi}{\partial\sigma_2}\right|_{W_2} \frac{d\sigma_2}{dW_2}$$

$$\text{----------------} \qquad (21)$$

$$\frac{\partial\phi}{\partial W_N} = \left.\frac{\partial\phi}{\partial W_N}\right|_{\sigma_N} + \left.\frac{\partial\phi}{\partial\sigma_N}\right|_{W_N} \frac{d\sigma_N}{dW_N}$$

Equations (19) then become

$$\frac{\partial P}{\partial W_1} + \lambda \left.\frac{\partial\phi}{\partial W_1}\right|_{\sigma_1} + \lambda \left.\frac{\partial\phi}{\partial\sigma_1}\right|_{W_1} \frac{d\sigma_1}{dW_1} = 0$$

$$\frac{\partial P}{\partial W_2} + \lambda \left.\frac{\partial\phi}{\partial W_2}\right|_{\sigma_2} + \lambda \left.\frac{\partial\phi}{\partial\sigma_2}\right|_{W_2} \frac{d\sigma_2}{dW_2} = 0$$

$$\text{----------------} \qquad (22)$$

$$\frac{\partial P}{\partial W_N} + \lambda \left.\frac{\partial\phi}{\partial W_N}\right|_{\sigma_N} + \lambda \left.\frac{\partial\phi}{\partial\sigma_N}\right|_{W_N} \frac{d\sigma_N}{dW_N} = 0$$

and either P or ϕ constant.

When no variation of structural factor with stage weights is considered, the corresponding expressions for "optimization" can easily be shown* to be

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*See Appendix

$$\frac{\partial P}{\partial W_1} + \lambda \frac{\partial \phi}{\partial W_1}\bigg|_{\sigma_1} = 0$$

$$\frac{\partial P}{\partial W_2} + \lambda \frac{\partial \phi}{\partial W_2}\bigg|_{\sigma_2} = 0 \tag{23}$$

$$\frac{\partial P}{\partial W_N} + \lambda \frac{\partial \phi}{\partial W_N}\bigg|_{\sigma_N} = 0$$

and either P or ϕ = constant.

Since relations (22) are the true optimization conditions then, in order for relations (23) to be optimum, (22) and (23) must be compatible. It is clear that, in order for (22) and (23) to be compatible, the relations

$$\frac{\partial \phi}{\partial \sigma_1}\bigg|_{W_1} \frac{d\sigma_1}{dW_1} = 0$$

$$\frac{\partial \phi}{\partial \sigma_2}\bigg|_{W_2} \frac{d\sigma_2}{dW_2} = 0 \tag{24}$$

$$\frac{\partial \phi}{\partial \sigma_N}\bigg|_{W_N} \frac{d\sigma_N}{dW_N} = 0$$

must hold.

Since (24) cannot be valid, except in the nonrealistic case where the structural factor does not vary with stage weight, Eqs. (22) and (23) are not compatible. Hence, Eqs. (23) do not represent a true optimization criterion for realistic rocket design. The actual discrepancy in the design criterion in using (23) rather than (22) will depend upon the relative magnitudes of the terms in (22) that have the coefficient λ. If the second term of coefficient λ in (22) is negligible compared to the first term of coefficient λ, then (23) represents a realistic optimization. In general, however, this is not the case.

D. Reduction of the Optimization Equations to Simpler Form

From Eq. (15), for fixed W_L:

$$\frac{\partial P}{\partial W_1} = \frac{\partial P}{\partial W_2} = \ldots = \frac{\partial P}{\partial W_N} = \frac{1}{W_L} \tag{25}$$

Substituting (25) into (19) and eliminating common terms, the optimization equations reduce to:

$$\frac{\partial \phi}{\partial W_1} = \frac{\partial \phi}{\partial W_2} = \cdots = \frac{\partial \phi}{\partial W_N} \tag{26}$$

and either ϕ or P = constant

IV. APPLICATION TO OPTIMUM THREE-STAGE ROCKET DESIGN

For a three-stage rocket, Eqs. (15) and (16) become

$$P = \frac{W_1 + W_2 + W_3 + W_L}{W_L} \tag{27}$$

and

$$\phi = \left(\frac{W_1 + W_2 + W_3 + W_L}{C_1 W_1^{n_1} + W_2 + W_3 + W_L}\right)^{I_1} \left(\frac{W_2 + W_3 + W_L}{C_2 W_2^{n_2} + W_3 + W_L}\right)^{I_2} \left(\frac{W_3 + W_L}{C_3 W_3^{n_3} + W_L}\right)^{I_3} \tag{28}$$

Accordingly,

$$\frac{\partial \phi}{\partial W_1} = \frac{\phi I_1 \left[(1 - n_1) C_1 W_1^{n_1} + \left(1 - n_1 C_1 W_1^{n_1 - 1}\right)(W_2 + W_3 + W_L)\right]}{\left(W_1 + W_2 + W_3 + W_L\right)\left(C_1 W_1^{n_1} + W_2 + W_3 + W_L\right)} \tag{29}$$

$$\frac{\partial \phi}{\partial W_2} = \frac{\phi I_1 \left(C_1 W_1^{n_1 - 1} - 1\right) W_1}{\left(W_1 + W_2 + W_3 + W_L\right)\left(C_1 W_1^{n_1} + W_2 + W_3 + W_L\right)} + \frac{\phi I_2 \left[(1 - n_2) C_2 W_2^{n_2} + \left(1 - n_2 C_2 W_2^{n_2 - 1}\right)(W_3 + W_L)\right]}{\left(W_2 + W_3 + W_L\right)\left(C_2 W_2^{n_2} + W_3 + W_L\right)} \tag{30}$$

and

$$\frac{\partial \phi}{\partial W_3} = \frac{\phi I_1 \left(C_1 W_1^{n_1-1} - 1\right) W_1}{\left(W_1 + W_2 + W_3 + W_L\right)\left(C_1 W_1^{n_1} + W_2 + W_3 + W_L\right)}$$

$$+ \frac{\phi I_2 \left(C_2 W_2^{n_2-1} - 1\right) W_2}{\left(W_2 + W_3 + W_L\right)\left(C_2 W_2^{n_2} + W_3 + W_L\right)}$$

$$+ \frac{\phi I_3 \left[\left(1 - n_3\right) C_3 W_3^{n_3} + \left(1 - n_3 C_3 W_3^{n_3-1}\right) W_L\right]}{\left(W_3 + W_L\right)\left(C_3 W_3^{n_3} + W_L\right)} \qquad (31)$$

Substituting (29), (30), and (31) into the optimization equations,

$$\frac{\partial \phi}{\partial W_1} = \frac{\partial \phi}{\partial W_2} = \frac{\partial \phi}{\partial W_3} \ ,$$

and factoring out ϕ yields two independent nonlinear equations:

$$\frac{I_1 \left(1 - n_1 C_1 W_1^{n_1-1}\right)}{\left(1 - C_1 W_1^{n_1-1}\right)} \left[1 - C_1 W_1^{n_1-1} \left(\frac{W_1 + W_2 + W_3 + W_L}{C_1 W_1^{n_1} + W_2 + W_3 + W_L}\right)\right]$$

$$= I_2 \left[1 - n_2 C_2 W_2^{n_2-1} \left(\frac{W_2 + W_3 + W_L}{C_2 W_2^{n_2} + W_3 + W_L}\right)\right] \qquad (32)$$

$$I_2 \frac{\left(1-n_2C_2W_2^{n_2-1}\right)}{\left(1-C_2W_2^{n_2-1}\right)}\left[1-C_2W_2^{n_2-1}\left(\frac{W_2+W_3+W_L}{C_2W_2^{n_2}+W_3+W_L}\right)\right]$$

$$= I_3\left[1-n_3C_3W_3^{n_3-1}\left(\frac{W_3+W_L}{C_3W_3^{n_3}+W_L}\right)\right] \tag{33}$$

Equations (32) and (33) together with the constraint equation, ϕ or P = constant, constitute three equations in terms of the three unknowns W_1, W_2, and W_3. For specified sets of values of I_1, I_2, I_3, C_1, C_2, C_3, n_1, n_2, n_3, and ϕ or P (depending upon which type of optimization is desired), the three equations can be solved by iteration techniques to yield optimum values of W_1, and W_2, and W_3. By employing Equations (11) and (13), the other rocket parameters can also be determined.

V. APPLICATION TO OPTIMIZING TWO-STAGE ROCKET DESIGN

Analogous to the three stage optimization, the optimization conditions for a two stage rocket become:

$$\frac{\partial \phi}{\partial W_1} = \frac{\partial \phi}{\partial W_2}$$

$$\text{and} \quad \phi \text{ or } P = \text{constant} \tag{34}$$

which yield:

$$I_1 \frac{\left(1 - n_1 C_1 W_1^{n_1 - 1}\right)}{\left(1 - C_1 W_1^{n_1 - 1}\right)} \left[1 - C_1 W_1^{n_1 - 1} \left(\frac{W_1 + W_2 + W_L}{C_1 W_1^{n_1} + W_2 + W_L}\right)\right]$$

$$= I_2 \left[1 - n_2 C_2 W_2^{n_2 - 1} \left(\frac{W_2 + W_L}{C_2 W_2^{n_2} + W_L}\right)\right] \tag{35}$$

Equation (35) together with the contraint, ϕ or P = constant, can be solved by iteration to yield optimal values of W_1 and W_2.

VI. APPLICATION TO N-STAGE ROCKET OPTIMIZATION

If iterative techniques can be expanded to include solving N simultaneous nonlinear equations, then the process described above can be extended to any number of stages. In the case of a large number of stages, computerized random search techniques might be employed to solve for optimum values of the stage weights. Once the optimum stage weights are obtained, these values can be substituted into Eqs. (11) and (13) to solve for the other rocket parameters.

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APPENDIX

Further comparison with previous methods writing ϕ for a three stage rocket in terms of σ_1, σ_2, and σ_3,

$$\phi = \left(\frac{W_1 + W_2 + W_3 + W_L}{\sigma_1 W_1 + W_2 + W_3 + W_L}\right)^{I_1} \left(\frac{W_2 + W_3 + W_L}{\sigma_2 W_2 + W_3 + W_L}\right)^{I_2} \left(\frac{W_3 + W_L}{\sigma_3 W_3 + W_L}\right)^{I_3} \quad (36)$$

Then

$$\left.\frac{\partial \phi}{\partial W_1}\right|_{\sigma_1} = \frac{I_1 \phi (1 - \sigma_1)(W_2 + W_3 + W_L)}{(\sigma_1 W_1 + W_2 + W_3 + W_L)(W_1 + W_2 + W_3 + W_L)} \quad (37)$$

$$\left.\frac{\partial \phi}{\partial W_2}\right|_{\sigma_2} = \frac{I_1 \phi (\sigma_1 - 1) W_1}{(\sigma_1 W_1 + W_2 + W_3 + W_L)(W_1 + W_2 + W_3 + W_L)} + \frac{I_2 \phi (1 - \sigma_2)(W_3 + W_L)}{(\sigma_2 W_2 + W_3 + W_L)(W_2 + W_3 + W_L)} \quad (38)$$

and

$$\left.\frac{\partial \phi}{\partial W_3}\right|_{\sigma_3} = \frac{I_1 \phi (\sigma_1 - 1) W_1}{(\sigma_1 W_1 + W_2 + W_3 + W_L)(W_1 + W_2 + W_3 + W_L)} + \frac{I_2 \phi (\sigma_2 - 1) W_2}{(\sigma_2 W_2 + W_3 + W_L)(W_2 + W_3 + W_L)} + \frac{I_3 \phi (1 - \sigma_3) W_L}{(\sigma_3 W_3 + W_L)(W_3 + W_L)} \quad (39)$$

Substuting (37), (38) and (39) into the conditions from (23):

$$\left.\frac{\partial \phi}{\partial W_1}\right|_{\sigma_1} = \left.\frac{\partial \phi}{\partial W_2}\right|_{\sigma_2} = \left.\frac{\partial \phi}{\partial W_3}\right|_{\sigma_3}$$

result in:

$$I_1\left[1 - \sigma_1\left(\frac{W_1 + W_2 + W_3 + W_L}{\sigma_1 W_1 + W_2 + W_3 + W_L}\right)\right] = I_2\left[1 - \sigma_2\left(\frac{W_2 + W_3 + W_L}{\sigma_2 W_2 + W_3 + W_L}\right)\right]$$

$$= I_3\left[1 - \sigma_3\left(\frac{W_3 + W_L}{\sigma_3 W_3 + W_L}\right)\right] \qquad (40)$$

Equation (40) represents the relations corresponding to previous methods (1-11) for design criteria. By using Eqs. (11) and (13), then Eqs. (40), (32), and (33) can be written for comparison purposes in terms of the mass ratios. Equation (40) then becomes

$$I_1(1 - \sigma_1 r_1) = I_2(1 - \sigma_2 r_2) = I_3(1 - \sigma_3 r_3) \qquad (41)$$

which is the more familiar form usually seen in the references listed above. Equations (32) and (33) become:

$$I_1 \frac{(1 - n_1 \sigma_1)}{(1 - \sigma_1)} (1 - \sigma_1 r_1) = I_2(1 - n_2 \sigma_2 r_2) \qquad (42)$$

$$I_2 \frac{(1 - n_2 \sigma_2)}{(1 - \sigma_2)} (1 - \sigma_2 r_2) = I_3(1 - n_3 \sigma_3 r_3) \qquad (43)$$

or combining (42) and (43):

$$I_1 \frac{(1 - n_1 \sigma_1)}{(1 - \sigma_1)} (1 - \sigma_1 r_1) = I_2(1 - n_2 \sigma_2 r_2) = I_2(1 - n_2) + n_2 \frac{I_3(1 - \sigma_2)}{(1 - n_2 \sigma_2)} \times$$

$$(1 - n_3 \sigma_3 r_3) \qquad (44)$$

which represent more inclusive optimization conditions. Comparison of (41) with (44) indicates the significant difference between previous "optimization" and the more realistic optimization criteria.