Resonant cavities and space-time

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Abstract

I analyze the behavior of electromagnetic fields inside a cavity by solving Einstein field equations. It is shown that the modified geometry of space-time inside the cavity due to a propagating mode can affect the propagation of a laser beam. The effect is the appearance of components of laser light with a shifted frequency originating from the coupling between the laser field and the mode cavity due to gravity. The analysis is extended to the case of a frustum taken to be a truncated cone. It is shown that a proper choice of the geometrical parameters of the cavity can make the gravitational effect significant.

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I. INTRODUCTION

A single plane wave always induces a deformation of the geometry of the space-time [1]. This effect is so small and plane waves such idealized objects that hopes to observe it are certainly very tiny. Anyway, electromagnetic fields are easily available and the technology is old and it is not impossible to realize devices where the intensity of such fields could make this gravitational effect observable. This entails a rather sensible interferometer but it is not impossible to realize. The devices that better fit the aim are resonant cavities where, due to large merit factors, gross intensity of the electromagnetic energy can be achieved. The observation of such an effect would mean a real breakthrough in experimental general relativity as, so far, only large scale measurements were considered possible and so available. A table-top experiment would completely change our way to manage gravitational fields and could pave the way to a possible engineering of space-time due to our ability to manage and produce electromagnetic fields.

In this paper I show a simple textbook computation in general relativity showing how a resonant cavity with a single mode excited inside could provide a satsfactory set-up for such a measurement and could explain the recent measurements done at Eagleworks using a frustum as a resonator. This should be considered just the starting point for a more extended treatment to the experimental set-up, much on the same lines of Ref.[2]. In the latter paper a modified Einstein theory was considered but, with proper adjustments, the computations could easily fit the bill in our case.

Finally, I analize the case of a cavity having the form of a truncated cone, a frustum. I show that, with a proper choice of the geometrical parameters of the cavity, gravitational effects could be enhanced. This could explain some recent measurements performed at Eagleworks labs of NASA.

II. PLANE WAVE GEOMETRY

A. Geometry

The simplest case discussed in literature for the Einstein-Maxwell equations is that of a plane wave [1]. I take the metric in the form

$$ds^2 = L^2(v)(dx^2 + dy^2) - dvdu \tag{1}$$

given the Rosen coordinates $v = ct - z$ and $u = ct + z$. It is easy to show that an electromagnetic plane wave modifies the geometry of space-time. I have that the Einstein tensor reduces to the Ricci tensor as the trace of the energy-momentum tensor is zero in this case. I will have the only non-null component

$$R_{33} = -2\frac{L''(v)}{L(v)}. \tag{2}$$

The electromagnetic field tensor will have the non-null components

$$F_{31} = -F_{13} = A'(v). \tag{3}$$

So, the only nonzero component of the energy-momentum tensor is

$$T_{33} = -\frac{1}{\mu_0}\frac{(A'(v))^2}{L^2(v)} \tag{4}$$

and so I have to solve the equation

$$2\frac{L''(v)}{L(v)} = -\frac{8\pi G}{c^4\mu_0}\frac{(A'(v))^2}{L^2(v)} \tag{5}$$

That has the solution $L(v) = \pm\alpha A'(v)$ provided

$$\alpha^2 = \frac{4\pi G}{c^2\omega^2\mu_0} \tag{6}$$

and I am left with the equation for a plane wave

$$[A'(v)]'' + \frac{\omega^2}{c^2}A'(v) = 0 \tag{7}$$

for the electromagnetic field and taking $A'(0) = E_0/c$ the magnetic field amplitude. Note that $\alpha \approx 9 \cdot 10^{-21}\ A \cdot m \cdot N^{-1} = 9 \cdot 10^{-21}\ T^{-1}$ for $\omega = 1\ GHz$. This is a small number as expected and this effect is negligible small for all practical purposes. Its inverse identify a critical magnetic field for which this effect could be meaningful but has an unphysical large value.

In a resonant cavity, an estimation of the amplitude of the electric field E_0 can be computed using the formula [2]

$$\frac{\epsilon_0}{4}E_0^2L^3 = \frac{Q\cdot P}{\omega} \tag{8}$$

being Q the merit factor, P the input power and V the volume of the cavity assumed to be a box of side length L. In this case I have to apply the boundary condition

$$A'(0) = A'(L). \tag{9}$$

This yields the modes to be $k_n = 2n\pi/L$, being n an integer, and the corresponding frequencies $\omega_n = ck_n$ arising from the Rosen coordinate $v = ct - z$.

B. Light propagation

I assume that a beam of light is moving through the box containing the mode described above as the cavity is fed through some source. There is no electromagnetic interaction between these two electromagnetic fields because light has not self-interaction besides a small effect, dubbed Delbrück scattering, that can be analyzed in quantum electrodynamics and is fourth order. This competes with the gravitational correction. The propagation of the beam inside the cavity is described by the wave equation

$$L^2(v)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right) - 4\frac{\partial^2\psi}{\partial u\partial v} = 0 \tag{10}$$

and one sees that the altered geometry by the mode of the cavity can couple it with the laser beam. This equation can be solved by separation of variables setting

$$\psi(x,y,u,v) = \mathcal{E}(x,y)\phi(u,v) \tag{11}$$

being $\mathcal{E}(x,y)$ an envelope of the beam. This yields the equation for $\phi(u,v)$

$$-4\frac{\partial^2\phi}{\partial u\partial v} = k^2L^2(v)\phi \tag{12}$$

that is

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial z^2} = k^2L^2(ct-z)\phi. \tag{13}$$

One can consider $L(ct-z)$ a small quantity and do some perturbation theory yielding

$$\phi(z,t) \approx \phi_0(z,t) + \frac{ck^2}{2}\int dz'dt'\theta(c(t-t')-(z-z'))L^2(ct'-z')\phi_0(z',t') \tag{14}$$

being $\theta(z)$ the Heaviside step function and $\phi_0(z,t)$ the laser beam entering the cavity. Finally, one has

$$\psi(x,y,z,t) \approx \psi_0(x,y,z,t) + \frac{ck^2}{2}\int dz'dt'\theta(c(t-t')-(z-z'))L^2(ct'-z')\psi_0(x,y,z',t'). \tag{15}$$

One sees that there is an additional component to the laser field exiting the cavity that interacts with the mode inside. This can have terms with the frequency shifted and is a purely gravitational effect. In order to see this just note that

$$L^2(ct-z) = \alpha^2\frac{E_0^2}{2c^2}\left(2+e^{i\omega(t-z/c)}+e^{-i\omega(t-z/c)}\right) \tag{16}$$

and, for the laser field,

$$\psi_0(x,y,z,t) = A(x,y,z)e^{i\omega_L t} + A^*(x,y,z)e^{-i\omega_L t}. \tag{17}$$

Putting this into eq.(15) one sees that the additional components contribute as

$$\psi(x,y,z,t) \approx \psi_0(x,y,z,t) + k^2\alpha^2\frac{E_0^2}{4c}\left(A_1(x,y,z)e^{i\omega_L t} + A_2(x,y,z)e^{i(\omega-\omega_L)t} + A_3(x,y,z)e^{i(\omega+\omega_L)t} + c.c.\right). \tag{18}$$

One should observe satellite lines due to the modified geometry of space-time originating from the field inside the cavity. Note also the dependence on k that for a laser can be very large and one gets an overall noticeable effect.

III. FRUSTUM CASE

We assume now a new set of coordinates (r,θ,ϕ,ct) and the geometry has a rotational symmetry along the z axis. The geometrical form is that of a truncated cone dubbed frustum

A. Modes

If the cavity has the form of a frustum, the modes inside take the form [2]

$$\mathbf{B} = -U_0 k R(r) S'(\theta)\cos(\omega t)\,\mathbf{e}_\varphi, \tag{19}$$

$$\mathbf{E}/c = U_0\left\{\frac{R(r)}{r}n(n+1)S(\theta)\,\mathbf{e}_r + \left[\frac{R(r)}{r}+R'(r)\right]S'(\theta)\,\mathbf{e}_\theta\right\}\sin(\omega t) \tag{20}$$

where U_0 is a global constant dependent on the source supplying the cavity and the characteristics of the cavity itself. The functions R and S are defined as

$$
\begin{aligned}
S(\theta) &= P_n(\cos\theta), \\
R(r) &= R_+(r)\cos\alpha + R_-(r)\sin\alpha, \\
R_\pm(r) &= \frac{J_{\pm(n+1/2)}(kr)}{\sqrt{r}},
\end{aligned}
$$

where P_n is the Legendre polynomial of order n, J_m the Bessel function of the first kind of order m, and α and k constants to be determined along with the order n. By boundary conditions, the order n of the Legendre polynomial must satisfy

$$P_n(\cos\theta_0) = 0,$$

being θ_0 the semi-angle of the cone, the wavenumber k the condition

$$\left[\frac{R_+}{r} + R'_+\right]_{r_2}\left[\frac{R_-}{r} + R'_-\right]_{r_1} = \left[\frac{R_+}{r} + R'_+\right]_{r_1}\left[\frac{R_-}{r} + R'_-\right]_{r_2},$$

and α

$$\tan\alpha = -\frac{R_+(r_2)/r_2 + R'_+(r_2)}{R_-(r_2)/r_2 + R'_-(r_2)}.$$

The resonant mode angular frequency is thus determined as $\omega = kc$. From this we can compute the non-zero components of the energy-momentum tensor of the electromagnetic field. We get

$$
\begin{aligned}
F_{10} &= -F_{01} = U_0\frac{R(r)}{r}n(n+1)S(\theta)\sin(\omega t) \\
F_{20} &= -F_{02} = \left[\frac{R(r)}{r} + R'(r)\right]S'(\theta)\sin(\omega t) \\
F_{32} &= -F_{23} = -U_0 k R(r) S'(\theta)\cos(\omega t)
\end{aligned}
\tag{21}
$$

The constant U_0 can be obtained using the formula [2]

$$\frac{\int\langle B^2\rangle\, dV}{\mu_0} = \frac{U_0^2k^2}{2\mu_0}\int[R(r)S'(\theta)]^2\, dV = \frac{QP}{\omega}, \tag{22}$$

being Q the quality factor of the cavity, P the input power and a time average is applied.

IV. EINSTEIN EQUATIONS

It is not difficult to realize that the quantity, taken V the volume of the cavity,

$$\kappa = \frac{8\pi G}{c^4} \approx 2.0765\cdot 10^{-43}N^{-1} \tag{23}$$

is small and so we have to eventually apply the linearized theory. There is no way out of this unless something else is at work that physicists have not yet accounted for in general relativity. The Equations equations in the Donder gauge are [3]

$$-\frac{1}{2}\Box\overline{h}_{\mu\nu} = \kappa(T_{\mu\nu} + \tau_{\mu\nu}) \tag{24}$$

being $\tau_{\mu\nu}$ the gravity stress-energy tensor and

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}. \tag{25}$$

We have set at the start

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{26}$$

being $\eta_{\mu\nu}$ the flat metric and $h_{\mu\nu}$ the gravity field. This is not a tensor but it is not a concern here. We work out the analysis as given in [4]. We get the general solution

$$\overline{h}_{\mu\nu}(\boldsymbol{x}, t) = -2\kappa \int_V d^3x' \frac{(T_{\mu\nu} + \tau_{\mu\nu})\left(\boldsymbol{x}', t - \frac{\boldsymbol{x}-\boldsymbol{x}'}{c}\right)}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{27}$$

being $\tau_{\mu\nu}$ the Landau-Lifshitz pseudotensor of the gravity field. We introduce the constant

$$l_0^{-2} = 2\kappa\frac{U_0^2}{\mu_0} \approx 3.3 \cdot 10^{-37}U_0^2 \tag{28}$$

with U_0^2 given in T^2 and being the definition of a length. This means that

$$\overline{h}_{\mu\nu}(\boldsymbol{x}, t) = -l_0^{-2} \int_V d^3x' \frac{(\overline{T}_{\mu\nu} + \mu_0 U_0^{-2}\tau_{\mu\nu})\left(\boldsymbol{x}', t - \frac{\boldsymbol{x}-\boldsymbol{x}'}{c}\right)}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{29}$$

being $\overline{T}_{\mu\nu}$ the dimensionless energy-momentum tensor of the electromagnetic field inside the cavity. We can remove l_0 by changing the length scale in the integral and obtain

$$\overline{h}_{\mu\nu}(\overline{\boldsymbol{x}}, \overline{t}) = -\int_V d^3\overline{x}' \frac{(\overline{T}_{\mu\nu} + 2\kappa\overline{\tau}_{\mu\nu})\left(\overline{\boldsymbol{x}}', \overline{t} - \frac{\overline{\boldsymbol{x}}-\overline{\boldsymbol{x}}'}{c}\right)}{|\overline{\boldsymbol{x}} - \overline{\boldsymbol{x}}'|} \tag{30}$$

having set $\overline{\boldsymbol{x}} = \boldsymbol{x}/l_0$ and $\overline{t} = t/(l_0/c)$. $\overline{\tau}_{\mu\nu}$ is the normalized gravity pseudotensor. This has a prefactor $(2\kappa)^{-1}$. l_0 is really large unless we are in the field of a magnetar. Then, the integral is easy to evaluate to give

$$\overline{h}_{\mu\nu}(\overline{\boldsymbol{x}}, \overline{t}) = -L(\overline{\boldsymbol{x}})\left(\overline{T}_{\mu\nu} + 2\kappa\overline{\tau}_{\mu\nu}\right)\left(\overline{\boldsymbol{x}}, \overline{t}\right) \tag{31}$$

being

$$L(\overline{\boldsymbol{x}}) = \int_V d^3\overline{x}' \frac{1}{|\overline{\boldsymbol{x}} - \overline{\boldsymbol{x}}'|} \tag{32}$$

a geometrical factor obtained by integrating on the volume of the frustum. Eq.(31) would be a differential equation for $\overline{h}_{\mu\nu}$ but, in a first approximation, we can assume that the derivatives of it are negligible and we are left with the result

$$\overline{h}_{\mu\nu}(\overline{\boldsymbol{x}},\overline{t}) = -L(\overline{\boldsymbol{x}})\overline{T}_{\mu\nu}\left(\overline{\boldsymbol{x}},\overline{t}\right). \tag{33}$$

This is our key result and can be stated in the same way as inductance enters into electromagnetic field.

V. GRAVITATIONAL SUSCEPTIBILITY OF THE FRUSTUM

The susceptibility of the frustum can be evaluated by computing the integral

$$L(r,z,\theta) = \int_0^h dz' \int_0^{2\pi} d\theta' \int_0^{\frac{r_2-r_1}{h}z'+r_2} r'dr' \frac{1}{\sqrt{r^2+r'^2+(z-z')^2-2rr'\cos(\theta-\theta')}} \tag{34}$$

that is rather involved. A way out is to note that

$$\Delta_2 \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{35}$$

and so

$$\Delta_2 L(\overline{\boldsymbol{x}}) = 1. \tag{36}$$

The solution of this equation is

$$L(\overline{\boldsymbol{x}}) = L_o(\overline{\boldsymbol{x}}) + a + b\ln(\overline{r}) + \frac{\overline{r}^2}{4} \tag{37}$$

being $L_o(\overline{\boldsymbol{x}})$ a solution of the equation $\Delta_2 L_o(\overline{\boldsymbol{x}}) = 0$, we assume it to be zero, and we have to set the condition for the frustum

$$\overline{r}(\overline{z}) = \frac{r_2-r_1}{h}\overline{z} + \frac{r_1}{l_0}. \tag{38}$$

This yields

$$\begin{aligned} a &= \ln^{-1}\frac{r_2}{r_1}\left(\frac{1}{4}\frac{r_2^2}{l_0^2}\ln\frac{r_1}{l_0} - \frac{1}{4}\frac{r_1^2}{l_0^2}\ln\frac{r_2}{l_0}\right) \\ b &= \ln^{-1}\frac{r_2}{r_1}\left(\frac{1}{4}\frac{r_1^2}{l_0^2} - \frac{1}{4}\frac{r_2^2}{l_0^2}\right). \end{aligned} \tag{39}$$

These equations appear rather interesting as, by a proper choice of parameters, one can make a gravitational effect more or less relevant in the physics of the problem. *It is the case to say that geometry comes to rescue.*

VI. CONCLUSIONS

I have shown how a plane wave could produce a gravitational effect inside a cavity that could be observed using a propagating laser beam inside it. The effect could be unveiled using an interferometer or observing the components of the laser field outside the cavity. Components with a shifted frequency, due to the modes inside the cavity, should be seen. This could explain recent results at Eagleworks with a resonator having the form of a truncated cone. A local warp of the geometry due to the electromagnetic field pumped inside the cavity could be a satisfactory explanation. From a physical standpoint this could be a really breakthrough paving the way to table-top experiments in general relativity and marking the starting point of space-time engineering.

Then, I considered a frustum in the form of a truncated cone. I have shown that general relativity introduce a large scale that makes all the effects rather miniscule. But this can be overcome by a wise choice of the geometry. For the frustum I have shown that the gravitational effects can be described by a susceptibility multiplying the energy-momentum tensor of the electromagnetic field inside the cavity. Due to this particular geometry, it can be shown that the susceptibility can be made significant by a proper choice of the geometrical parameters of the cavity. This could explain some recent experimental results at Eagleworks but some prudence is needed. Anyhow, it is not difficult to put at test the equations we proposed.We hope to see such a test in the very near future.

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, "Gravitation," San Francisco 1973, 1279p.

[2] F. O. Minotti, Grav. Cosmol. **19**, 201 (2013) [arXiv:1302.5690 [gr-qc]].

[3] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980), pp. 320ff.

[4] S. Weinberg, "Gravitation and Cosmolgy", (Wiley, New York, 1972), pp. 165ff.